(b) Define another norm function $\|\cdot\|: C[a,b] \longrightarrow \mathbb{R}$ as

$$\|x\| = \int_a^b |x(t)|\,dt \quad \forall x \in C[a,b]$$

verify that $\|\cdot\|$ is a norm on $C[a,b]$.

(i) $\|x\| \geq 0$ $\quad \because |x(t)| \geq 0 \quad \forall t \in [a,b]$

$$\Rightarrow \int_a^b |x(t)|dt \geq 0$$

$$\Rightarrow \|x\| \geq 0 \quad \forall x \in C[a,b].$$

(ii) $\|x\| = 0 \Leftrightarrow \int_a^b |x(t)|\,dt = 0$

$$\Leftrightarrow |x(t)| = 0, \quad \forall t \in [a,b]$$

$$\Leftrightarrow x = 0$$

(iii) $\|\alpha x\| = \int_a^b |(\alpha x)(t)|\,dt = \int_a^b |\alpha||x(t)|\,dt$

$$= |\alpha| \int_a^b |x(t)|\,dt$$

$$= |\alpha|\|x\| \quad \forall \alpha \in \mathbb{R} \text{ \& } x \in C[a,b]$$

(iv) Let $x, y \in C[a,b]$

$$\|x+y\| = \int_a^b |(x+y)(t)|\,dt$$

$$= \int_a^b |x(t)+y(t)|\,dt$$

$$\leq \int_a^b (|x(t)|+|y(t)|)\,dt$$

$$= \int_a^b |x(t)|dt + \int_a^b |y(t)|\,dt = \|x\|+\|y\|.$$

i.e $\|x+y\| \leq \|x\|+\|y\|$

$\therefore (C[a,b], \|\cdot\|)$ is a normed space (NOT Banach?)

<u>Note</u>: (*) A normed space has a completion which is a Banach space.

(*) A mapping from a normed space X into a normed space Y is called an operator.

(*) A mapping from a normed space X into the scalar field $\mathbb{R}$ or $\mathbb{F}$ is called functional.

We denote operator by capital letter (preferably T) and the image of an x under T by Tx, and functional by lower case letter (preferably f) and the value of f at an x by f(x).

(*) In the sequence space l^p. If $p=2$, then we call it the "Hilbert sequence space l^2."

* **Some inequalities**:-

(I) The Cauchy's Schwarz inequality for sum,

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{1/2}.$$